

THE SPECTRAL DENSITY OF THE SCATTERING MATRIX FOR HIGH ENERGIES

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ABSTRACT. We determine the density of eigenvalues of the scattering matrix of the Schrödinger operator with a short range potential in the high energy asymptotic regime. We give an explicit formula for this density in terms of the X-ray transform of the potential.

1. INTRODUCTION AND MAIN RESULT

1.1. Introduction. The object of study in this paper is the (on-shell) scattering matrix $S(\lambda)$ corresponding to the scattering of a d -dimensional quantum particle on an external short range potential V at the energy $\lambda > 0$. The scattering matrix $S(\lambda)$ is a unitary operator in $L^2(\mathbb{S}^{d-1})$ and the difference $S(\lambda) - I$ is compact. Thus, the eigenvalues of $S(\lambda)$ can be written as

$$(1.1) \quad \exp(i\theta_n(\lambda)), \quad \theta_n(\lambda) \in [-\pi, \pi), \quad n \in \mathbb{N}$$

and $\theta_n(\lambda) \rightarrow 0$ as $n \rightarrow \infty$. The quantities $\theta_n(\lambda)$ are known as scattering phases. The scattering phases are usually discussed in physics literature (see e.g. [6, Section 123]) under the additional assumption of the spherical symmetry of V ; here we do not need this assumption.

Our aim is to study the asymptotic distribution of scattering phases $\{\theta_n(\lambda)\}_{n=1}^\infty$ when $\lambda \rightarrow \infty$. It turns out that after an appropriate scaling the asymptotic density of scattering phases can be described by a simple explicit formula involving the X-ray transform (see (1.5)) of the potential V . This formula has a semiclassical nature.

The key idea of this paper goes back to the work of M. Sh. Birman and D. R. Yafaev [1] (see also [2]), where the asymptotics of $\theta_n(\lambda)$ for a *fixed* λ and $n \rightarrow \infty$ was determined for some class of potentials V . This asymptotic behaviour is *not* uniform in λ , and thus our results cannot be derived from those of [1]. However, both the results of [1] and our results are based on the following observation. The leading term of the asymptotics of $\theta_n(\lambda)$ (in both asymptotic regimes) is determined by the Born approximation of the scattering matrix. The Born approximation is essentially a pseudodifferential operator (Ψ DO) on the sphere \mathbb{S}^{d-1} with the symbol given by the X-ray transform of V . Standard Ψ DO results can be used to give spectral asymptotics for operators of such type. In both asymptotic regimes, the desired spectral asymptotics are given by a Weyl type formula involving the symbol of the Ψ DO.

1.2. The scattering matrix. Let us briefly recall the relevant definitions. Let $H_0 = -\Delta$ and $H = -\Delta + V$ be the Schrödinger operators in $L^2(\mathbb{R}^d)$, $d \geq 2$, where V is the operator of

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multiplication by a real-valued potential $V \in C(\mathbb{R}^d)$ which is assumed to satisfy the estimate

$$(1.2) \quad |V(x)| \leq \frac{C}{(1+|x|)^\rho}, \quad \rho > 1$$

with some constant $C > 0$. It is one of the fundamental facts of scattering theory [4, 5] that under the assumption (1.2) the wave operators

$$W_\pm = \text{s-lim}_{t \rightarrow \pm\infty} e^{itH} e^{-itH_0}$$

exist and are complete; the scattering operator $\mathbf{S} = W_+^* W_-$ is unitary in $L^2(\mathbb{R}^d)$ and commutes with H_0 . Let $F : L^2(\mathbb{R}^d) \rightarrow L^2((0, \infty); L^2(\mathbb{S}^{d-1}))$ be the unitary operator

$$(Fu)(\lambda, \omega) = \frac{1}{\sqrt{2}} \lambda^{(d-2)/4} \widehat{u}(\sqrt{\lambda}\omega), \quad \lambda > 0, \quad \omega \in \mathbb{S}^{d-1},$$

where \widehat{u} is the usual (unitary) Fourier transform of u . The operator F diagonalises H_0 , i.e.

$$(FH_0 u)(\lambda, \omega) = \lambda(Fu)(\lambda, \omega), \quad \forall u \in C_0^\infty(\mathbb{R}^d).$$

The commutation relation $\mathbf{S}H_0 = H_0\mathbf{S}$ implies that F also diagonalises \mathbf{S} , i.e.

$$(F\mathbf{S}u)(\lambda, \cdot) = S(\lambda)(Fu)(\lambda, \cdot),$$

where $S(\lambda) : L^2(\mathbb{S}^{d-1}) \rightarrow L^2(\mathbb{S}^{d-1})$ is the unitary operator known as the (on-shell) scattering matrix. See e.g. the book [9] for the details.

Under the assumption (1.2) the operator $S(\lambda) - I$ is compact and consequently the eigenvalues of $S(\lambda)$ (enumerated with multiplicities taken into account) can be written as (1.1) with $\theta_n(\lambda) \rightarrow 0$ as $n \rightarrow \infty$.

1.3. The purpose of the paper. Our purpose is to describe the asymptotic density of the eigenvalues of $S(\lambda)$ as $\lambda \rightarrow \infty$. We recall the estimate (see e.g. [9, Section 8.1])

$$(1.3) \quad \|S(\lambda) - I\| = O(\lambda^{-1/2}), \quad \lambda \rightarrow \infty.$$

This estimate is sharp, i.e. $O(\lambda^{-1/2})$ cannot be replaced by $o(\lambda^{-1/2})$; this can be seen by considering the case of a spherically symmetric potential and using the separation of variables. Thus, the spectrum of the scattering matrix $S(\lambda)$ for large λ consists of a cluster of eigenvalues located on an arc of length $O(\lambda^{-1/2})$ around 1. Let us define the eigenvalue counting measure for $S(\lambda)$. The estimate (1.3) suggests the following scaling: for $\lambda \geq 1$, set $k = \sqrt{\lambda} > 0$ and define (using notation (1.1))

$$(1.4) \quad \mu_k([\alpha, \beta]) = \#\{n \in \mathbb{N} : \alpha \leq k\theta_n(k^2) \leq \beta\}, \quad [\alpha, \beta] \subset \mathbb{R} \setminus \{0\}$$

where $\#$ denotes the number of elements in the set. We will study the weak asymptotics of μ_k as $k \rightarrow \infty$, i.e. we consider the asymptotics of the integrals

$$\int_{-\infty}^{\infty} \psi(t) d\mu_k(t), \quad k \rightarrow \infty$$

for test functions $\psi \in C_0^\infty(\mathbb{R} \setminus \{0\})$.

1.4. Main result. In order to describe the weak limit of the measures μ_k , we need to fix some notation. For any $\omega \in \mathbb{S}^{d-1}$, let $\Lambda_\omega \subset \mathbb{R}^d$ denote the hyperplane passing through the origin and orthogonal to ω . We equip both \mathbb{S}^{d-1} and Λ_ω with the standard $(d-1)$ -dimensional Lebesgue measure (=Euclidean area). We set

$$(1.5) \quad X(\omega, \eta) = -\frac{1}{2} \int_{-\infty}^{\infty} V(t\omega + \eta) dt, \quad \omega \in \mathbb{S}^{d-1}, \quad \eta \in \Lambda_\omega.$$

The function X (up to a multiplicative factor) is known as the X-ray transform of V in the inverse problem literature. The following elementary estimate is a direct consequence of (1.2):

$$(1.6) \quad |X(\omega, \eta)| \leq C(V)(1 + |\eta|)^{1-\rho}, \quad \omega \in \mathbb{S}^{d-1}, \quad \eta \in \Lambda_\omega$$

with some constant $C(V)$. We define a measure μ on $\mathbb{R} \setminus \{0\}$ by

$$\mu([\alpha, \beta]) = (2\pi)^{1-d} \text{meas}\{(\omega, \eta) \in \mathbb{S}^{d-1} \times \Lambda_\omega : \alpha \leq X(\omega, \eta) \leq \beta\}, \quad [\alpha, \beta] \subset \mathbb{R} \setminus \{0\},$$

where meas denotes the usual product measure. By the boundedness of V , the measure μ has a compact support. The measure μ need not be absolutely continuous. The measure μ may be weakly singular at zero in the following sense: $\mu((0, \infty))$ or $\mu((-\infty, 0))$ may be infinite, but, by the estimate (1.6) we have

$$(1.7) \quad \int_{-\infty}^{\infty} |t|^\ell d\mu(t) < \infty, \quad \forall \ell > (d-1)/(\rho-1).$$

Our main result is as follows:

Theorem 1.1. *Let $V \in C(\mathbb{R}^d)$ be a potential satisfying (1.2). Then for any test function $\psi \in C_0^\infty(\mathbb{R} \setminus \{0\})$,*

$$(1.8) \quad \lim_{k \rightarrow \infty} k^{1-d} \int_{-\infty}^{\infty} \psi(t) d\mu_k(t) = \int_{-\infty}^{\infty} \psi(t) d\mu(t).$$

This can be more succinctly put as the weak convergence of the measures

$$(1.9) \quad k^{1-d} \mu_k \rightarrow \mu, \quad k \rightarrow \infty.$$

Much of the inspiration for both the content of this paper and the proofs may be found in [7], where similar asymptotics are determined for the spectrum of the Landau Hamiltonian (i.e. two-dimensional Schrödinger operator with a constant homogeneous magnetic field) perturbed by a potential which obeys the same condition (1.2).

We would like to mention also the paper [10] where the high energy asymptotic distribution of the phases $\theta_n(\lambda)$ was studied for scattering problems on manifolds of a certain special class. The results of [10] are much more detailed than ours and include the asymptotics of the pair correlation measure.

1.5. Comparison with [1]. In [1], the case of potentials with the power asymptotics at infinity of the type

$$(1.10) \quad V(x) = v(x/|x|)|x|^{-\rho}(1 + o(1)), \quad |x| \rightarrow \infty, \quad \rho > 1,$$

was considered. Using our notation μ_k, μ , the result of [1] can be written as

$$(1.11) \quad \begin{aligned} k^{1-d} \mu_k((\varepsilon, \infty)) &\sim \mu((\varepsilon, \infty)), \\ k^{1-d} \mu_k((-\infty, -\varepsilon)) &\sim \mu((-\infty, -\varepsilon)), \end{aligned}$$

when $k > 0$ is fixed and $\varepsilon \rightarrow +0$. Here $a \sim b$ means $\frac{a}{b} \rightarrow 1$.

Clearly, our main result (1.9) is expressed by the same formula as (1.11), but the asymptotic regimes are different. Neither of the results (1.9), (1.11) implies the other one.

1.6. Semiclassical interpretation. By the definition of the scattering operator \mathbf{S} , for any $\psi \in L^2(\mathbb{R}^d)$ we have

$$\begin{aligned} i((\mathbf{S} - I)\psi, \psi) &= i \lim_{t \rightarrow \infty} ((e^{-2itH} e^{itH_0} \psi, e^{-itH_0} \psi) - \|\psi\|^2) = i \int_0^\infty \frac{d}{dt} (e^{-2itH} e^{itH_0} \psi, e^{-itH_0} \psi) dt \\ &= \int_0^\infty (V e^{-2itH} e^{itH_0} \psi, e^{-itH_0} \psi) dt + \int_0^\infty (V e^{itH_0} \psi, e^{2itH} e^{-itH_0} \psi) dt. \end{aligned}$$

If ψ corresponds to large energies, the right hand side can be approximated by the first term in its expansion in powers of V . This means that we can replace e^{itH} by e^{itH_0} in the above expressions, and so

$$(1.12) \quad i((\mathbf{S} - I)\psi, \psi) \approx \int_{-\infty}^\infty (V e^{-itH_0} \psi, e^{-itH_0} \psi) dt, \quad \psi \in L^2(\mathbb{R}^d),$$

which is exactly the Born approximation in the time-dependent picture.

In order to write down the classical analogue of the right hand side of (1.12), assume that ψ is concentrated near x in the coordinate representation and near p in the momentum representation. Then ψ represents a particle with the coordinate x and momentum p , and in the same way $e^{-itH_0} \psi$ represents a particle with the coordinate $x + 2pt$ and momentum p . Thus, the classical analogue of the right hand side of (1.12) is

$$\int_{-\infty}^\infty V(x + 2pt) dt = \frac{1}{2|p|} \int_{-\infty}^\infty V(x + \omega t') dt', \quad (x, p) \in \mathbb{R}^d \times \mathbb{R}^d,$$

where $\omega = \frac{p}{|p|} \in \mathbb{S}^{d-1}$. This calculation explains the appearance of the X-ray transform in the asymptotics of $S(\lambda)$.

1.7. Key steps of the proof. First we recall the stationary representation for the scattering matrix. For $k > 0$ and $\rho > 1$, we define the operator $\Gamma_\rho(k) : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{S}^{d-1})$ by

$$(1.13) \quad (\Gamma_\rho(k)u)(\omega) = \frac{1}{\sqrt{2}} k^{(d-2)/2} (2\pi)^{-d/2} \int_{\mathbb{R}^d} u(x) \langle x \rangle^{-\rho/2} e^{-ik\langle x, \omega \rangle} dx, \quad \omega \in \mathbb{S}^{d-1},$$

where $\langle x \rangle = (1 + |x|^2)^{1/2}$. By the Sobolev trace theorem, $\Gamma_\rho(k)$ is bounded for $\rho > 1$. Next, let

$$T(z) = \langle x \rangle^{-\rho/2} (H - zI)^{-1} \langle x \rangle^{-\rho/2}, \quad \text{Im } z > 0;$$

according to the limiting absorption principle, the limits $T(k^2 \pm i0)$ exist in the operator norm for all $k > 0$. Denote by J the bounded operator of multiplication by $\langle x \rangle^\rho V(x)$ in $L^2(\mathbb{R}^d)$. The stationary representation for the scattering matrix can be written as (see e.g. [9, Section 6.6])

$$S(k^2) = I - 2\pi i \Gamma_\rho(k) (J - JT(k^2 + i0)J) \Gamma_\rho(k)^*, \quad k > 0.$$

The asymptotic density of eigenvalues of $S(k^2)$ for large k is determined by the *Born approximation* of the scattering matrix, defined as

$$(1.14) \quad S_B(k^2) = I - 2\pi i \Gamma_\rho(k) J \Gamma_\rho(k)^*, \quad k > 0.$$

The key observation due to M. Birman and D. Yafaev [1] is that the operator $\Gamma_\rho(k)J\Gamma_\rho(k)^*$ in $L^2(\mathbb{S}^{d-1})$ with the integral kernel

$$2^{-1}k^{d-2}(2\pi)^{-d} \int_{\mathbb{R}^d} e^{-ik\langle\omega-\omega',x\rangle} V(x)dx, \quad \omega, \omega' \in \mathbb{S}^{d-1}$$

can be represented as a Ψ DO on the sphere with the symbol (up to inessential constants) $X(\omega, \eta)$. We combine this observation with the standard semiclassical pseudodifferential techniques to obtain the spectral asymptotics of $S_B(k^2)$. In this way we prove the asymptotic formula (see Lemma 3.2)

$$(1.15) \quad \lim_{k \rightarrow \infty} k^{1-d} \text{Tr}(\text{Im } kS_B(k^2))^\ell = \int_{-\infty}^{\infty} t^\ell d\mu(t)$$

for any natural number $\ell > (d-1)/(\rho-1)$; note that the r.h.s. of (1.15) is finite by (1.7).

Using (1.15) and the estimates for the Schatten norm of $S(k^2) - S_B(k^2)$ we prove that (1.8) holds true for test functions $\psi(t)$ which coincide with t^ℓ for all sufficiently small t . Theorem 1.1 follows by an application of the Weierstrass approximation theorem.

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2. PRELIMINARY STATEMENTS

2.1. The limiting absorption principle and its consequences. First we need some notation. We denote by S_p , $p \geq 1$, the usual Schatten class and by $\|\cdot\|_p$ the associated norm. Let X_ρ be the normed linear space of all potentials $V \in C(\mathbb{R}^d)$ satisfying (1.2) with the norm

$$\|V\|_{X_\rho} = \sup_{x \in \mathbb{R}^d} |V(x)| \langle x \rangle^\rho.$$

We recall the following estimates from [9]:

Proposition 2.1. *Let $V \in X_\rho$ with some $\rho > 1$. Then for any $\ell \geq 1$ satisfying $\ell > \frac{d-1}{\rho-1}$, one has*

$$(2.1) \quad \sup_{k \geq 1} k^{\frac{1-d}{\ell}} \|k \text{Im } S_B(k^2)\|_\ell \leq C(\ell, \rho, d) \|V\|_{X_\rho},$$

$$(2.2) \quad \sup_{k \geq 1} k^{\frac{1-d}{\ell}} \|k^2 \text{Im } (S_B(k^2) - S(k^2))\|_\ell \leq C(\ell, \rho, d, V),$$

$$(2.3) \quad \sup_{k \geq 1} k^{\frac{1-d}{\ell}} \|k \text{Im } S(k^2)\|_\ell \leq C(\ell, \rho, d, V).$$

The estimate (2.1) is a direct consequence of [9, Proposition 8.1.3]. The estimate (2.2) is proven in [9, Proposition 8.1.4]. The estimate (2.3) is a direct consequence of (2.1) and (2.2).

Lemma 2.2. *Let $V \in X_\rho$ with $\rho > 1$. Then for any integer $\ell \geq 1$ satisfying $\ell > \frac{d-1}{\rho-1}$, one has*

$$|\text{Tr}(k \text{Im } S(k^2))^\ell - \text{Tr}(k \text{Im } S_B(k^2))^\ell| = O(k^{d-2}), \quad k \rightarrow \infty.$$

Proof. From

$$A^\ell - B^\ell = \sum_{j=0}^{\ell-1} A^j (A - B) B^{\ell-1-j}$$

one easily obtains

$$(2.4) \quad |\mathrm{Tr}(A^\ell - B^\ell)| \leq \ell \|A - B\|_\ell \max\{\|A\|_\ell^{\ell-1}, \|B\|_\ell^{\ell-1}\}, \quad A, B \in S_\ell.$$

Thus, it suffices to prove the relation

$$\|k \mathrm{Im}(S(k^2) - S_B(k^2))\|_\ell \max\{\|k \mathrm{Im} S(k^2)\|_\ell^{\ell-1}, \|k \mathrm{Im} S_B(k^2)\|_\ell^{\ell-1}\} = O(k^{d-2}), \quad k \rightarrow \infty.$$

The latter relation follows by combining (2.1)–(2.3). \square

2.2. Semiclassical Ψ DO on the sphere. A semiclassical Ψ DO in $L^2(\mathbb{S}^{d-1})$ can be defined in a variety of ways; below we describe a slightly non-standard approach to this definition, which will simplify our exposition in Section 3.

For $\omega, \omega' \in \mathbb{S}^{d-1}$ such that $\omega + \omega' \neq 0$, we set

$$(2.5) \quad \kappa = \kappa(\omega, \omega') = \frac{\omega + \omega'}{|\omega + \omega'|} \in \mathbb{S}^{d-1}.$$

Clearly, $\kappa(\omega, \omega')$ is a smooth function of $(\omega, \omega') \in \mathbb{S}^{d-1} \times \mathbb{S}^{d-1}$ away from the anti-diagonal

$$\mathrm{AD} = \{(\omega, \omega') \in \mathbb{S}^{d-1} \times \mathbb{S}^{d-1} \mid \omega + \omega' = 0\}.$$

In order to overcome the (inessential) difficulties related to the singularity of κ at the anti-diagonal, we will assume that our amplitudes vanish in an open neighbourhood of AD. We will say that a function $b = b(\omega, \omega', \eta)$, $\omega, \omega' \in \mathbb{S}^{d-1}$, $\eta \in \Lambda_{\kappa(\omega, \omega')}$, is an *admissible amplitude*, if:

- b is a C^∞ -smooth function of its arguments;
- $b(\omega, \omega', \eta) = 0$ if $|\eta|$ is sufficiently large;
- $b(\omega, \omega', \eta) = 0$ if (ω, ω') are in an open neighbourhood of AD.

For an admissible amplitude b and a semiclassical parameter $k > 0$, we define the operator $\mathrm{Op}_k[b]$ in $L^2(\mathbb{S}^{d-1})$ via its integral kernel

$$(2.6) \quad \mathrm{Op}_k[b](\omega, \omega') = \left(\frac{k}{2\pi}\right)^{d-1} \int_{\Lambda_{\kappa(\omega, \omega')}} e^{-ik\langle \omega - \omega', \eta \rangle} b(\omega, \omega', \eta) d\eta,$$

where $\omega, \omega' \in \mathbb{S}^{d-1}$. It is easy to see that for $\omega \neq \omega'$ one has

$$\mathrm{Op}_k[b](\omega, \omega') = O(k^{-\infty}), \quad k \rightarrow \infty.$$

This shows that the values of the amplitude $b(\omega, \omega', \eta)$ away from an open neighbourhood of the diagonal $\omega = \omega'$ do not affect the asymptotic properties of the operator $\mathrm{Op}_k[b]$ as $k \rightarrow \infty$.

Proposition 2.3. *For any admissible amplitude b and any $k > 0$, the operator $\mathrm{Op}_k[b]$ is trace class, and for any $\ell \in \mathbb{N}$ one has*

$$\lim_{k \rightarrow \infty} \left(\frac{k}{2\pi}\right)^{-d+1} \mathrm{Tr}(\mathrm{Op}_k[b])^\ell = \int_{\mathbb{S}^{d-1}} \int_{\Lambda_\omega} b(\omega, \omega, \eta)^\ell d\eta d\omega.$$

Sketch of proof. The proof follows standard methods of Ψ DO theory; see e.g. [3, Theorem 9.6] for a similar statement in the context of operators in \mathbb{R}^n . Here we only outline the main steps.

First note that for $\ell = 1$ the result of Proposition 2.3 is trivial, since by a direct evaluation of trace we have the identity

$$(2.7) \quad \left(\frac{k}{2\pi}\right)^{-d+1} \text{Tr}(\text{Op}_k[b]) = \int_{\mathbb{S}^{d-1}} \int_{\Lambda_\omega} b(\omega, \omega, \eta) d\eta d\omega.$$

Next, using the local coordinates on the sphere and the composition formula for symbols of Ψ DOs (see e.g. [3, Proposition 7.7]), we obtain the following statement. For any $N > 0$ there exists $M > 0$ such that $(\text{Op}_k[b])^\ell$ can be represented as

$$(2.8) \quad (\text{Op}_k[b])^\ell = \sum_{m=0}^M k^{-m} \text{Op}_k[b_m] + R_k,$$

where b_m are admissible symbols, b_0 is such that

$$(2.9) \quad b_0(\omega, \omega, \eta) = b(\omega, \omega, \eta)^\ell, \quad \forall \omega \in \mathbb{S}^{d-1}, \quad \forall \eta \in \Lambda_\omega,$$

and R_k is an integral operator with a smooth kernel which satisfies

$$\sup_{\omega, \omega'} |R_k(\omega, \omega')| = O(k^{-N}), \quad k \rightarrow \infty.$$

Now taking $N > d - 1$ and evaluating the trace in (2.8), we get

$$\lim_{k \rightarrow \infty} \left(\frac{k}{2\pi}\right)^{-d+1} \text{Tr}(\text{Op}_k[b])^\ell = \lim_{k \rightarrow \infty} \left(\frac{k}{2\pi}\right)^{-d+1} \text{Tr}(\text{Op}_k[b_0]) = \int_{\mathbb{S}^{d-1}} \int_{\Lambda_\omega} b_0(\omega, \omega, \eta) d\eta d\omega.$$

In view of (2.9), this proves the required identity. \square

3. PROOF OF THEOREM 1.1

3.1. The Born approximation with $V \in C_0^\infty(\mathbb{R}^d)$.

Lemma 3.1. *Let $V \in C_0^\infty(\mathbb{R}^d)$. Then for any $\ell \in \mathbb{N}$, one has*

$$(3.1) \quad \lim_{k \rightarrow \infty} k^{1-d} \text{Tr}(k \text{Im } S_B(k^2))^\ell = \int_{-\infty}^{\infty} t^\ell d\mu(t).$$

Proof. 1. For ease of notation we write $Q(k) = k \text{Im } S_B(k^2)$. By (1.13) and (1.14), $Q(k)$ is the integral operator in $L^2(\mathbb{S}^{d-1})$ with the integral kernel

$$Q(k)(\omega, \omega') = -\frac{1}{2} \left(\frac{k}{2\pi}\right)^{d-1} \int_{\mathbb{R}^d} e^{-ik\langle \omega - \omega', x \rangle} V(x) dx, \quad \omega, \omega' \in \mathbb{S}^{d-1}.$$

For fixed $\omega, \omega', \omega + \omega' \neq 0$, let $\kappa = \kappa(\omega, \omega')$ be as in (2.5). Write any $x \in \mathbb{R}^d$ as $x = \kappa t + \eta$ with $t \in \mathbb{R}, \eta \in \Lambda_\kappa$. Note that by the orthogonality relation $(\omega - \omega') \perp \kappa$, one has

$$\langle \omega - \omega', x \rangle = \langle \omega - \omega', \eta \rangle.$$

Thus, the integral kernel of $Q(k)$ can be rewritten as

$$(3.2) \quad Q(k)(\omega, \omega') = -\frac{1}{2} \left(\frac{k}{2\pi} \right)^{d-1} \int_{\Lambda_{\kappa(\omega, \omega')}} \int_{-\infty}^{\infty} e^{-ik\langle \omega - \omega', \eta \rangle} V(\kappa(\omega, \omega')t + \eta) dt d\eta \\ = \left(\frac{k}{2\pi} \right)^{d-1} \int_{\Lambda_{\kappa(\omega, \omega')}} e^{-ik\langle \omega - \omega', \eta \rangle} X(\kappa(\omega, \omega'), \eta) d\eta,$$

where X is given by (1.5). From here we directly obtain the required identity (3.1) in the case $\ell = 1$ by integrating the kernel of $Q(k)$ over the diagonal. Now it remains to prove (3.1) for $\ell \geq 2$.

2. Let $\chi_0 \in C_0^\infty(\mathbb{S}^{d-1} \times \mathbb{S}^{d-1})$ be such that $\chi_0(\omega, \omega') = 1$ in an open neighbourhood of the diagonal $\omega = \omega'$ and $\chi_0(\omega, \omega') = 0$ in an open neighbourhood of the anti-diagonal $\omega + \omega' = 0$. Denote $\chi_1 = 1 - \chi_0$, and let

$$Q(k) = Q_0(k) + Q_1(k),$$

where $Q_j(k)$ is the operator with the integral kernel $\chi_j(\omega, \omega')Q(k)(\omega, \omega')$. By the fast decay of the Fourier transform of V and by the fact that $|\omega - \omega'|$ is separated away from zero on the support of χ_1 , we see that

$$\sup_{\omega, \omega'} |Q_1(k)(\omega, \omega')| = O(k^{-\infty}), \quad k \rightarrow \infty.$$

Thus, it suffices to prove that

$$\lim_{k \rightarrow \infty} k^{1-d} \text{Tr}(Q_0(k))^\ell = \int_{-\infty}^{\infty} t^\ell d\mu(t).$$

From (3.2) it follows that $Q_0(k)$ can be represented as a semiclassical Ψ DO on the sphere of the type (2.6):

$$Q_0(k) = \text{Op}_k[b], \quad \text{where} \quad b(\omega, \omega', \eta) = \chi_0(\omega, \omega')X(\kappa(\omega, \omega'), \eta),$$

b is an admissible amplitude in the sense discussed in Section 2.2, and X is given by (1.5). Applying Proposition 2.3, we get

$$(3.3) \quad \lim_{k \rightarrow \infty} k^{1-d} \text{Tr}(Q_0(k))^\ell = (2\pi)^{1-d} \int_{\mathbb{S}^{d-1}} \int_{\Lambda_\omega} X(\omega, \eta)^\ell d\eta d\omega = \int_{-\infty}^{\infty} t^\ell d\mu(t),$$

as required. \square

3.2. The Born approximation with general V .

Lemma 3.2. *Let $V \in X_\rho$ with $\rho > 1$. Then for any integer $\ell \geq 1$ satisfying $\ell > \frac{d-1}{\rho-1}$ one has*

$$\lim_{k \rightarrow \infty} k^{1-d} \text{Tr}(\text{Im } kS_B(k^2))^\ell = \int_{-\infty}^{\infty} t^\ell d\mu(t).$$

Proof. Let X_ρ^0 be the closure of $C_0^\infty(\mathbb{R}^d)$ in X_ρ . For any $\ell > \frac{d-1}{\rho-1}$, denote

$$g_\ell(V) = \int_{-\infty}^{\infty} t^\ell d\mu(t), \\ g_\ell^+(V) = \limsup_{k \rightarrow \infty} k^{1-d} \text{Tr}(k \text{Im } S_B(k^2))^\ell, \\ g_\ell^-(V) = \liminf_{k \rightarrow \infty} k^{1-d} \text{Tr}(k \text{Im } S_B(k^2))^\ell.$$

By Lemma 3.1, for all $V \in C_0^\infty(\mathbb{R}^d)$ we have

$$(3.4) \quad g_\ell(V) = g_\ell^+(V) = g_\ell^-(V).$$

Recall that $S_B(k^2)$ depends linearly on V . Using this fact and the estimates (2.1) and (2.4), it is easy to check that $g_\ell^\pm(V)$ are continuous functionals on X_ρ . Next, using the last equality in (3.3) and the estimate (1.6), it is easy to see that $g_\ell(V)$ is a continuous functional on X_ρ . Thus, by an approximation argument, (3.4) holds for any $V \in X_\rho^0$. Finally, for any $V \in X_\rho$ and a given $\ell > \frac{d-1}{\rho-1}$, choose ρ_1 such that $1 < \rho_1 < \rho$ with $\ell > \frac{d-1}{\rho_1-1}$. Then $X_\rho \subset X_{\rho_1}^0$ and the previous argument proves (3.4) for all $V \in X_{\rho_1}^0$ which suffices. \square

3.3. From the Born approximation to the full scattering matrix.

Lemma 3.3. *Let $V \in X_\rho$ with $\rho > 1$. Then for any integer $\ell \geq 1$ satisfying $\ell + 2 > \frac{d-1}{\rho-1}$,*

$$\lim_{k \rightarrow \infty} k^{1-d} \int_{-\infty}^{\infty} t^\ell d\mu_k(t) = \int_{-\infty}^{\infty} t^\ell d\mu(t).$$

Proof. By Lemmas 2.2 and 3.2, it suffices to prove

$$(3.5) \quad \lim_{k \rightarrow \infty} k^{1-d} \left| \int_{-\infty}^{\infty} t^\ell d\mu_k(t) - \text{Tr}(\text{Im } kS(k^2))^\ell \right| = 0.$$

Recalling the definition (1.4) of the measure μ_k , one sees that (3.5) is equivalent to

$$(3.6) \quad \lim_{k \rightarrow \infty} k^{1-d+\ell} \left| \sum_{n=1}^{\infty} [(\theta_n(k^2))^\ell - (\sin \theta_n(k^2))^\ell] \right| = 0.$$

By (1.3) we have $0 < |\theta_n(k^2)| < \pi/4$ for all sufficiently large k and all n . From the elementary estimates $|\theta_n| \leq 2|\sin \theta_n|$ and $|\theta_n - \sin \theta_n| \leq C|\sin \theta_n|^3$ which hold for $|\theta_n| < \pi/4$, it follows that for all sufficiently large k

$$\begin{aligned} k^{1-d+\ell} \sum_{n=1}^{\infty} |(\theta_n)^\ell - (\sin \theta_n)^\ell| &\leq k^{1-d+\ell} \sum_{n=1}^{\infty} \left(|\theta_n - \sin \theta_n| \sum_{j=0}^{\ell-1} |\theta_n|^j |\sin \theta_n|^{\ell-1-j} \right) \\ &\leq k^{1-d+\ell} C(\ell) \sum_{n=1}^{\infty} |\sin \theta_n|^{\ell+2} = k^{1-d-2} C(\ell) \|\text{Im } kS(k^2)\|_{\ell+2}^{\ell+2}. \end{aligned}$$

Now (3.6) follows by applying the estimate (2.3) for $\|\text{Im } kS(k^2)\|_{\ell+2}$ to the result just obtained. \square

Proof of Theorem 1.1. By the estimate (1.3), the supports of μ_k are bounded uniformly in $k \geq 1$. On the other hand, by the boundedness of V , the support of μ is also bounded. Thus, we may choose $T > 0$ such that

$$\text{supp } \mu \subset [-T, T] \quad \text{and} \quad \text{supp } \mu_k \subset [-T, T] \quad \text{for all } k \geq 1.$$

Next, fix $\psi \in C_0^\infty(\mathbb{R} \setminus \{0\})$, and let ℓ_0 be an even natural number satisfying $\ell_0 > \frac{d-1}{\rho-1}$. Since $\psi(t)$ vanishes near $t = 0$ by assumption, the function $\psi(t)/t^{\ell_0}$ is smooth. By the Weierstrass approximation theorem, for any $\varepsilon > 0$ there exists a polynomial $\psi_0(t)$ such that

$$|\psi(t)t^{-\ell_0} - \psi_0(t)| \leq \varepsilon, \quad \forall t \in [-T, T].$$

Denoting $\psi_{\pm}(t) = (\psi_0(t) \pm \varepsilon)t^{\ell_0}$, we obtain

$$(3.7) \quad \psi_-(t) \leq \psi(t) \leq \psi_+(t), \quad \forall t \in [-T, T],$$

$$(3.8) \quad \psi_+(t) - \psi_-(t) = 2\varepsilon t^{\ell_0}.$$

By (3.7), we get

$$(3.9) \quad \int_{-\infty}^{\infty} \psi_-(t) d\mu_k(t) \leq \int_{-\infty}^{\infty} \psi(t) d\mu_k(t) \leq \int_{-\infty}^{\infty} \psi_+(t) d\mu_k(t).$$

By construction, $\psi_{\pm}(t)$ are polynomials which involve powers t^m with $m \geq \ell_0$. Thus, we can apply Lemma 3.3 to (3.9), which yields

$$\begin{aligned} \limsup_{k \rightarrow \infty} k^{1-d} \int_{-\infty}^{\infty} \psi(t) d\mu_k(t) &\leq \int_{-\infty}^{\infty} \psi_+(t) d\mu(t), \\ \liminf_{k \rightarrow \infty} k^{1-d} \int_{-\infty}^{\infty} \psi(t) d\mu_k(t) &\geq \int_{-\infty}^{\infty} \psi_-(t) d\mu(t). \end{aligned}$$

On the other hand, by (3.7), (3.8),

$$\int_{-\infty}^{\infty} \psi_-(t) d\mu(t) \leq \int_{-\infty}^{\infty} \psi(t) d\mu(t) \leq \int_{-\infty}^{\infty} \psi_+(t) d\mu(t)$$

and

$$\int_{-\infty}^{\infty} \psi_+(t) d\mu(t) - \int_{-\infty}^{\infty} \psi_-(t) d\mu(t) = 2\varepsilon \int_{-\infty}^{\infty} t^{\ell_0} d\mu(t).$$

By (1.6), the integral in the right hand side of the last estimate is finite; denote this integral by C . Combining the above estimates, we obtain

$$\begin{aligned} \limsup_{k \rightarrow \infty} k^{1-d} \int_{-\infty}^{\infty} \psi(t) d\mu_k(t) &\leq \int_{-\infty}^{\infty} \psi(t) d\mu(t) + 2\varepsilon C, \\ \liminf_{k \rightarrow \infty} k^{1-d} \int_{-\infty}^{\infty} \psi(t) d\mu_k(t) &\geq \int_{-\infty}^{\infty} \psi(t) d\mu(t) - 2\varepsilon C. \end{aligned}$$

Since $\varepsilon > 0$ can be taken arbitrary small, we obtain the required statement. \square

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